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Dynamics of a Soft-Spin van Hemmen Model. I. Phase and Bifurcation Diagrams for Stationary Distributions

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The dynamics of a soft-spin version of the van Hemmen spin-glass model is considered in the thermodynamic limit. Phase and bifurcation diagrams for quenched distributions are given. Phase coexistence, metastability, and hysteretic phenomena are found.

KEY WORDS: van Hemmen spin glass; dynamics of soft spins; nonlinear Fokker–Planck equation.

1. INTRODUCTION

Several years ago, Desai and Zwanzig[(1)] and Dawson[(2)] studied the dynamics of a mean-field model having equilibrium phase transitions in the thermodynamic limit. These works were later extended to problems of self-synchronization of nonlinear oscillators,[(3)] the stochastic mean-field Brusselator,[(4)] and relaxation oscillations of charge-density waves.[(5)] The simplicity of the mean-field coupling makes it possible to find a closed equation for the evolution of the single-particle distribution function in the thermodynamic limit.[(1–5)] Bifurcations of the distribution function characterize the phase transitions between stationary distributions,[(1,2)] or dynamic bifurcations to stable time-dependent distributions.[(3–5)]

As explained in ref. 5, the key reduced equation for the single-particle distribution can be derived for mean-field models involving random parameters besides the external noise sources. This is the case

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of the charge-density waves[5] where the presence of noisy parameters qualitatively alters the phase transition. To learn more about the effects of the random parameters, we consider a simple modification of Desai and Zwanzig's model.[1] We find that in addition to the stationary distributions present in this model (corresponding to para and ferromagnetic phases), there are new stationary distributions described by additional order parameters (corresponding to spin-glass and mixed phases). New phase transitions between these phases appear. Phase coexistence, metastability, and hysteresis are possible.

The model we want to consider is related to van Hemmen's spin glasses,[6] whose statics was analyzed in refs. 6–8. Glauber dynamics of van Hemmen glasses was studied by Choy and Sherrington.[9] For a recent review on spin glasses see Binder and Young.[10] We consider a soft-spin version of the van Hemmen model whose dynamics is described by systems of stochastic differential equations. In the thermodynamic limit a reduced description for the single-particle distribution function is then available and will be analyzed here.

The statics of van Hemmen's model is exactly soluble in the thermodynamic limit. This model incorporates frustration due to competition between ferromagnetic and antiferromagnetic bonds in a way related to the RKKY interaction. To be more specific, van Hemmen[6] considers N Ising spins interacting via the Hamiltonian

$$H_N = -\frac{J_0}{N} \sum_{i,j} S_i S_j - \sum_{i,j} J_{ij} S_i S_j - h \sum_{i=1}^{N} S_i \tag{1.1}$$

Here sums are over all the spins, the term with $J_0 > 0$ is a ferromagnetic coupling, h is an external field, and J_{ij} incorporates the randomness

$$J_{ij} = \frac{J}{N} (\xi_i \eta_j + \xi_j \eta_i) \tag{1.2}$$

where the ξ's and the η's are independent, identically distributed random variables with the same even distribution around zero and a finite variance. In this paper, we will consider the simple case where ξ_i, η_j take on the values $+1$, -1 with probability 1/2. In this case the lower critical dimension is three (ref. 8, Section 5.5). Thus, qualitative features (phases and phase transitions) of a model (1.1) but with nearest neighbor couplings are preserved by the mean-field model.

In the thermodynamic limit $N \to \infty$, the free energy per spin corresponding to (1.1) can be found exactly. The properties of this mean-field model are conveniently described in terms of the following order parameters:

$$m_N = \frac{1}{N} \sum_{i=1}^{N} S_i, \qquad q_{1N} = \frac{1}{N} \sum_{i=1}^{N} \xi_i S_i$$
$$q_{2N} = \frac{1}{N} \sum_{j=1}^{N} \eta_j S_j \tag{1.3}$$

It can be shown[7] that the free energy per spin

$$f(T) = kT \lim_{N \to \infty} 1/N \ln \operatorname{Tr} \exp(-H_N/kT)$$

exists with probability one and it is independent of specific ξ and η realizations. One finds $-f(T)/kT$ by solving a maximum problem. The solutions of the corresponding Euler–Lagrange equations are the different phases. There are four such phases characterized by m and q, with

$$m_N \to m, \qquad q_{1N} \to q, \qquad q_{2N} \to q \qquad \text{as} \quad N \to \infty$$

(a) Paramagnetic ($m = 0 = q$).

(b) Ferromagnetic ($m \neq 0$, $q = 0$).

(c) Spin glass ($m = 0$, $q \neq 0$).

(d) Mixed ($m \neq 0$, $q \neq 0$).

Phase coexistence and stability questions have also been considered in ref. 7.

The soft-spin version of van Hemmen's model is described by the following system of stochastic equations:

$$\frac{dx_j}{dt} = \mu(1 - x_j^2)\, x_j + T^{1/2} w_j(t)$$
$$- \frac{1}{N} \sum_{k=1}^{N} (J_0 + J\xi_j \eta_k + J\xi_k \eta_j)(x_j - x_k), \qquad j = 1,..., N \tag{1.4}$$

Here $w_i(t)$ is a Gaussian zero-mean white noise of correlation

$$\langle w_i(t)\, w_j(t') \rangle = \delta_{ij} \delta(t - t')$$

The soft-spins x_i are real variables; when $\mu \to +\infty$, they asymptotically take on the values ± 1. The ξ_i and η_j in (1.4) are as in (1.2).

In ref. 5 we derived a general result for the one-particle probability density corresponding to the system (1.4): Suppose that at $t = 0$ the

N-particle probability density $P_N(t; x_1,..., x_N; \xi_1,..., \eta_N)$ is a product of one-particle densities and that the ξ_i, η_j are independent, identically-distributed random variables with common distribution function $\rho(\cdot)$ [that is, all the random variables $\xi_1,..., \xi_N$ and $\eta_1,..., \eta_N$ have the same distribution function ρ, $\rho(\xi_1),..., \rho(\eta_N)$]. Then the one-particle probability density obeys the following nonlinear Fokker–Planck equation (NLFPE):

$$\begin{aligned}
\partial_t p &= \frac{1}{2}\partial_x^2 p - \partial_x \left\{ \left[\left(A - \frac{\theta^2}{2} x^2 \right) x + \bar{x} + \alpha(\xi \overline{x\eta} + \eta \overline{x\xi}) \right] p \right\} \\
1 &= \int_{-\infty}^{\infty} p(t, x; \xi, \eta)\, dx \\
\bar{x} &= \langle x, p \rangle \equiv \int xp(t, x; \xi, \eta)\, dx\, d\rho(\xi)\, d\rho(\eta) \\
\overline{x\xi} &= \langle x\xi, p \rangle \\
\overline{x\eta} &= \langle x\eta, p \rangle
\end{aligned} \tag{1.5}$$

We have nondimensionalized x, t, and p so that

$$\begin{aligned}
J_0 t &\to t \\
(J_0/T)^{1/2} x &\to x \\
(T/J_0)^{1/2}\, p(t, x; \xi, \eta) &\to p(t, x; \xi, \eta) \\
A &= \mu/J_0 - 1 \\
\theta &= (2T\mu)^{1/2}/J_0 \\
\alpha &= J/J_0
\end{aligned}$$

In (1.5) the normalization condition directly follows from the one for the N-particle probability density. That $N^{-1} \sum_{j=1}^{N} x_j$ tends to $\bar{x}$ as in (1.5) was proved by Dawson[2] for the Desai–Zwanzig model. Our derivation in ref. 5 suggests (but does not prove) that a similar central limit result holds also in the presence of random parameters.

In this paper we analyze different stationary solutions of the NLFPE (1.5) which correspond to paramagnetic, ferromagnetic, spin-glass, and mixed phases. Stability of these phases is analyzed by means of a Liapunov function first proposed by Shiino[11] for problems without random parameters. Phase coexistence, metastability, and hysteresis cycles are then found. We will analyze these phenomena by means of Brownian simulation in a future publication.

2. QUENCHED EQUILIBRIA AND LIAPUNOV FUNCTION

2.1. Stationary Solutions of the NLFPE and Quenched Phases

To find the stationary solutions of the NLPFE, we solve (1.5) for a time-independent probability density such that p and $\partial_x p$ vanish as $|x| \to \infty$. The resulting density is a quenched equilibrium distribution:

$$\begin{aligned} p_0(x; \xi, \eta) &= \zeta(\xi, \eta)^{-1} \exp\left\{Ax^2 - \frac{\theta^2}{4}x^4 + 2\bar{x}x + 2\alpha x(\xi\overline{x\eta} + \eta\overline{x\xi})\right\} \\ \zeta(\xi, \eta) &= \int_{-\infty}^{\infty} dx \exp\left\{Ax^2 - \frac{\theta^2}{4}x^4 + 2\bar{x}x + 2\alpha x(\xi\overline{x\eta} + \eta\overline{x\xi})\right\} \end{aligned} \tag{2.1}$$

For a quenched equilibrium to exist, the following consistency conditions must hold:

$$\begin{aligned} \bar{x} &= \langle x, p_0 \rangle \\ \overline{x\xi} &= \langle x\xi, p_0 \rangle \\ \overline{x\eta} &= \langle x\eta, p_0 \rangle \end{aligned} \tag{2.2}$$

That $\langle \ln \zeta, p_0 \rangle$ is a convex function of $\bar{x}$, $\overline{x\xi}$, and $\overline{x\eta}$ implies $\overline{x\xi} = \overline{x\eta} = q$. The proof is the same as that in ref. 7, pp. 321–322. Thus, we have stationary distributions described by two order parameters $\bar{x}$ and q and three dimensionless control parameters A, θ, and α. As in van Hemmen's model, we distinguish paramagnetic ($x = 0, q = 0$), ferromagnetic ($x \neq 0$, $q = 0$), spin-glass ($x = 0, q \neq 0$), and mixed ($x \neq 0, q \neq 0$) phases.

For the rest of this section we restrict ourselves to the following simple two-valued distribution for the (identically distributed) noises ξ and η:

$$d\rho(\xi) = \tfrac{1}{2}[\delta(\xi + 1) + \delta(\xi - 1)]\, d\xi \tag{2.3}$$

Then, Eqs. (2.2) become

$$\begin{aligned} \bar{x} &= \tfrac{1}{4}[m(2\bar{x} + 4\alpha q) + 2m(2\bar{x}) + m(2\bar{x} - 4\alpha q)] \\ q &= \tfrac{1}{4}[m(2\bar{x} + 4\alpha q) - m(2\bar{x} - 4\alpha q)] \end{aligned}$$

where

$$m(y) = \int_{-\infty}^{\infty} x \exp\left(Ax^2 - \frac{\theta^2}{4}x^4 + xy\right) dx \bigg/ \int_{-\infty}^{\infty} \exp\left(Ax^2 - \frac{\theta^2}{4}x^4 + xy\right) dx \tag{2.4}$$

In Fig. 1, we have displayed the phase diagram in the A, θ plane for $\alpha = 2$. Other values $\alpha > 1$ yield qualitatively the same diagram. From (2.1) and (2.4) we see that $q = 0$ if $\alpha = J/J_0 = 0$ (i.e., no randomness) and the NLFPE (1.5) becomes that of the Desai and Zwanzig.[1] Then, there is only a paramagnetic/ferromagnetic phase transition, whose locus in the (A, θ) plane we will call the Desai–Zwanzig (DZ) line. Since α always appears multiplying q in the right sides of (2.4), and $q = 0$ for both para and ferromagnetic phases, the DZ line remains unchanged for $\alpha \neq 0$.

There are four basic regions in Fig. 1. In region 1, the only stable stationary solution corresponds to the paramagnetic phase. We shall explain later how to decide whether a given solution is stable. In region 2, the paramagnetic phase has become unstable and a stable spin-glass phase has branched from it. An unstable ferromagnetic solution bifurcates from the paramagnetic solution at the DZ line. In region 3, there is coexistence between a mixed phase (which sprouts from the spin-glass phase) and the ferromagnetic solution, which has become stable. Finally, in the narrow region between 2 and 3, spin-glass and ferromagnetic phases coexist.

What happens when different coexistence lines are crossed is displayed in the bifurcation diagram of Fig. 2. There solid lines represent stable stationary solutions of (1.5), dashed lines correspond to the unstable stationary solutions, and dot-dashed lines are projections on the (q, θ) and

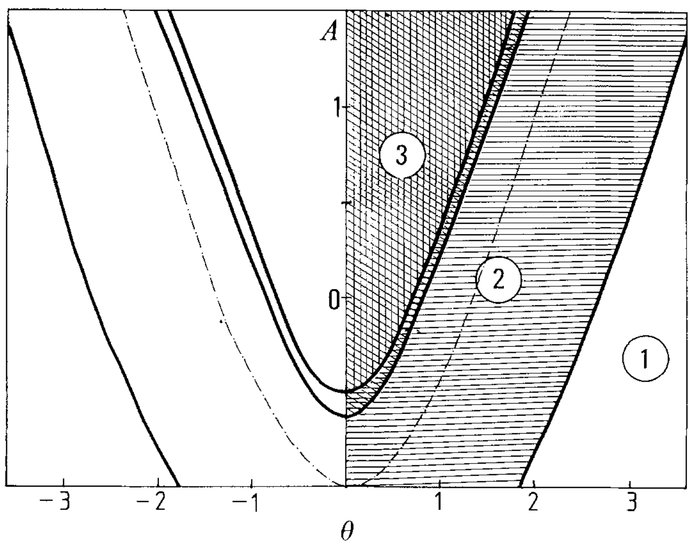

Fig. 1. Phase diagram in the plane $A = \mu/J_0 - 1$, $\theta = (2T\mu)^{1/2} J_0$ for $\alpha \equiv J/J_0 = 2$. The stable phases in each region are discussed in the text. The dot-dash line is the DZ line. The diagram is symmetric with respect to the A axis.

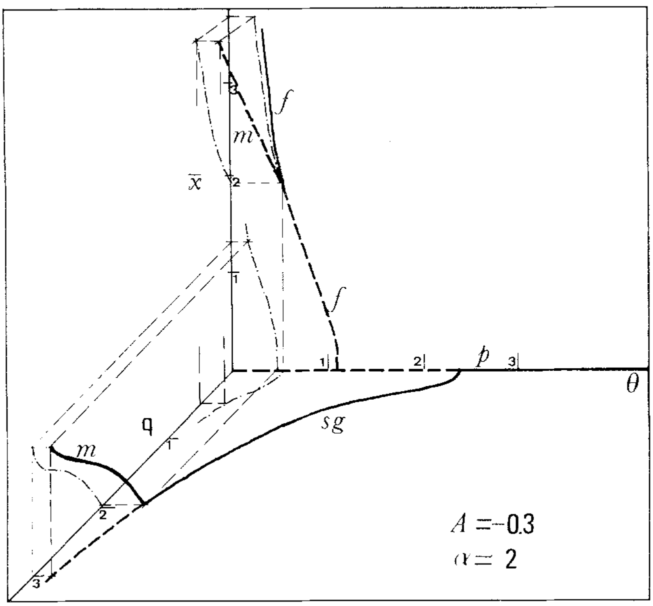

Fig. 2. Bifurcation diagram for the quenched case $A=-0.3$, $\alpha=2$, showing the different stable (solid lines) and unstable (dashed lines) stationary solutions. Only the octant $\theta \geqslant 0$, $\bar{x} \geqslant 0$, $q \geqslant 0$ is displayed because the whole diagram is symmetric with respect to the three axes.

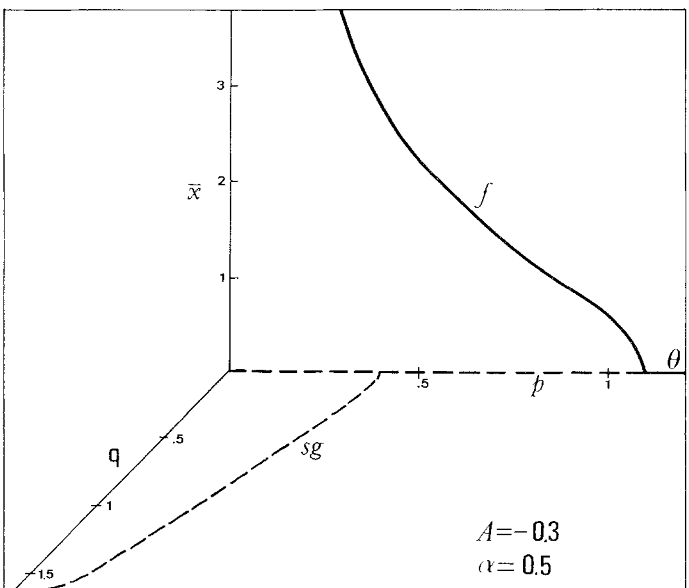

Fig. 3. Same as Fig. 2, for $\alpha=0.5$.

$(\bar{x}, \theta)$ planes of the mixed solutions. Given the even symmetry of the problem, we have represented the first octant of the $(q, \bar{x}, \theta)$ space only.

For $\alpha < 1$ the bifurcation diagram is depicted in Fig. 3. The only stable solutions corresponds to para and ferromagnetic phases. There is a branch of spin-glass solutions which is always unstable. The only relevant phase transition line is the DZ line. This means that the level of the disorder noise (1.2) has to surpass a certain threshold $J = J_0$ for the spin-glass and mixed phases to stabilize. Comparison with the results for van Hemmen's model (ref. 7, Sections 7 and 8; notice $\alpha = J_0/J$ in this reference) shows the bifurcation diagrams to be alike. Of course, the softness of our spins implies that $\bar{x}$ does not saturate as it does for hard spins.

2.2. Liapunov Function

In Figs. 2 and 3 we have used a Liapunov function to decide whether a given stationary solution is stable. It is

$$H[p(t, \cdot)] = \int p(t, x; \xi, \eta) \ln \frac{p(t, x; \xi, \eta)}{Q[p(t, \cdot)]} \, dx \, d\rho(\xi) \, d\rho(\eta) \tag{2.5}$$

$$Q[p(t, \cdot)] = \exp\left\{Ax^2 - \frac{\theta^2}{4} x^4 + 2x[\bar{x} + \alpha(\eta\overline{x\xi} + \xi\overline{x\eta})] - \bar{x}^2 - 2\alpha\, \overline{x\xi}\, \overline{x\eta}\right\} \tag{2.6}$$

The Liapunov function (2.5) is similar to one used by Shiino[11] for the Desai and Zwanzig[1] model.

To show that (2.5) is a Liapunov function, we have the prove that H is bounded below and that it is a nonincreasing function of t. The second property follows from direct algebraic manipulations:

$$\frac{d}{dt} H[p(t, \cdot)] = -\frac{1}{2} \langle p, (\partial_x \ln p/Q)^2 \rangle \leqslant 0 \tag{2.7}$$

where we have used

$$\left\langle p, \frac{d}{dt} \ln Q \right\rangle = 0 \tag{2.8}$$

To bound H, we use the inequality $x \ln x \geqslant x - 1$, $x \geqslant 0$, in the definition of H:

$$H = \langle Q, (p/Q) \ln(p/Q) \rangle \geqslant \langle Q, p/Q \rangle - \langle Q, 1 \rangle = 1 - \langle Q, 1 \rangle$$

Here

$$Q = \exp\left\{-(x-\bar{x})^2 - \alpha(\overline{x\xi} - x\eta)^2 - \alpha(\overline{x\eta} - x\xi)^2 + \alpha(\overline{x\xi} - \overline{\xi\eta})^2 + [A + 1 + \alpha(\xi^2 + \eta^2)]\, x^2 - \frac{\theta^2}{4} x^4\right\}$$

Therefore,

$$H \geqslant 1 - \exp[\alpha(\overline{x\xi} - \overline{x\eta})^2] \int \exp\left\{[A + 1 + \alpha(\xi^2 + \eta^2)]\, x^2 - \frac{\theta^2}{4} x^4\right\} dx\, d\rho(\xi)\, d\rho(\eta)$$

For the noise (2.3),

$$H \geqslant 1 - 4\exp[\alpha(\overline{x\xi} - \overline{x\eta})^2] \int_{-\infty}^{\infty} \exp\left[(A + 1 + 2\alpha)\, x^2 - \frac{\theta^2}{4} x^4\right] dx$$

which is finite for $\theta \neq 0$.

From our proof it follows that local minima of H yields stable solutions of the NLFPE (1.5). Restricting ourselves to the stationary solutions (2.1) (for which $\overline{x\xi} = \overline{x\eta} = q$ because of the convexity of $\langle p_0, \ln \zeta\rangle$, as stated before)

$$H_0 = H[p_0] = \left\langle p_0, \ln \frac{p_0}{Q_0}\right\rangle = \left\langle p_0, \ln \frac{\exp(\bar{x}^2 + 2\alpha q^2)}{\zeta}\right\rangle$$

i.e.,

$$H_0 = \bar{x}^2 + 2\alpha q^2 - \langle p_0, \ln \zeta\rangle = \bar{x}^2 + 2\alpha q^2 - \int d\rho(\xi)\, d\rho(\eta) \ln \zeta(\xi, \eta) \quad (2.9)$$

We can now show the following statement.

The condition that H_0 have a local minimum at $\underline{v}^* \equiv (\bar{x}, (2\alpha)^{1/2} q)$ is equivalent to $\underline{v}^*$ being a linearly stable fixed point of the equation

$$\frac{1}{2}\frac{\partial}{\partial v_j} \langle p_0, \ln \zeta\rangle = v_j, \qquad j = 1, 2 \quad (2.10)$$

This is also equivalent to the matrix $\varphi_{ij} \equiv \frac{1}{2}\partial^2 \langle p_0, \ln \zeta\rangle / \partial v_i\, \partial v_j$ having eigenvalues λ_i with $|\lambda_i| < 1$.

In fact, $\langle p_0, \ln \zeta\rangle$ is a convex function of v_1 and v_2,[7] and therefore φ_{ij} is a nonnegative matrix. That $\underline{v}^*$ is a linearly stable fixed point of (2.10) means the following: Let us start with a trial $\underline{y}$ sufficiently close to $\underline{y}^*$ and solve (2.10) by iteration. $\underline{y}^*$ is a stable fixed point if $\underline{y} \to \underline{y}^*$ in the interation scheme. Clearly $\underline{y}^*$ is a linearly stable fixed point of (2.10) if and only if the absolute value of each eigenvalue of φ_{ij} is less than one. In turn, the last statement is equivalent to $\delta_{ij} - \varphi_{ij}$ being strictly positive, as the variational characterization of the eigenvalues show. But $\delta_{ij} - \varphi_{ij}$ is the Hessian matrix corresponding to (2.10), evaluated at $\underline{y}^*$. Thus $\delta_{ij} - \varphi_{ij} > 0$ implies that $\frac{1}{2}H_0$ has a minimum at $\underline{y}^*$. QED

To decide the stability of a given branch of stationary solutions in Fig. 2 and 3, we have numerically evaluated the eigenvalues of φ_{ij} with the results displayed there.

The Liapunov function may in principle be used to visualize the basin of attraction of a given stationary solution. In Figs. 4 and 5 we have depicted the level curves of H_0 in (2.9) as a function of $\bar{x}$ and q. Minima of H_0 correspond to stationary solutions of (2.1). To any initial probability density $p(0, x; \xi, \eta)$ with $\overline{x\xi} = \overline{x\eta} = q$ there corresponds a point in the $\bar{x}$, q plane. We conjecture that if this point is in the basin of a given minimum in Figs. 4 and 5, the probability density will evolve toward the stationary density represented by that minimum. This conjecture is true for initial conditions close to the stationary probability densities and near

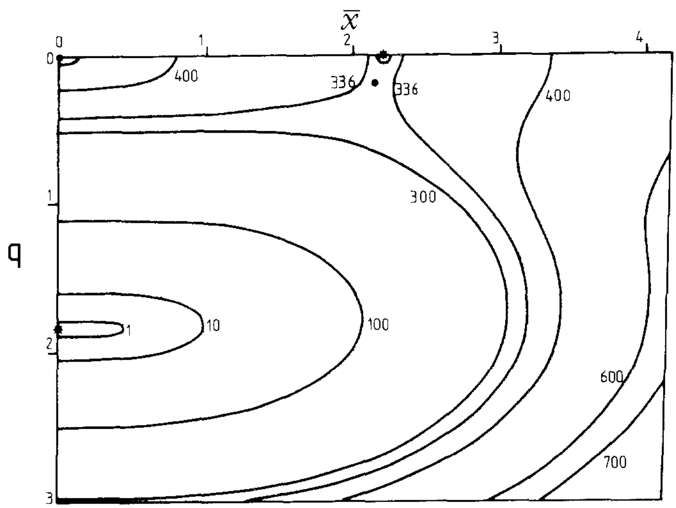

Fig. 4. Liapunov function for $\alpha = 2$, $\theta = 0.5$, $A = -0.3$, displaying coexistence between spin-glass and ferromagnetic phases. Relative minima are marked with a star and unstable stationary solutions with a dot. Numerical values are given to the curves so that the deepest minimum has the value zero and the absolute maximum has the value 999. The whole diagram is symmetrical with respect to both axes.

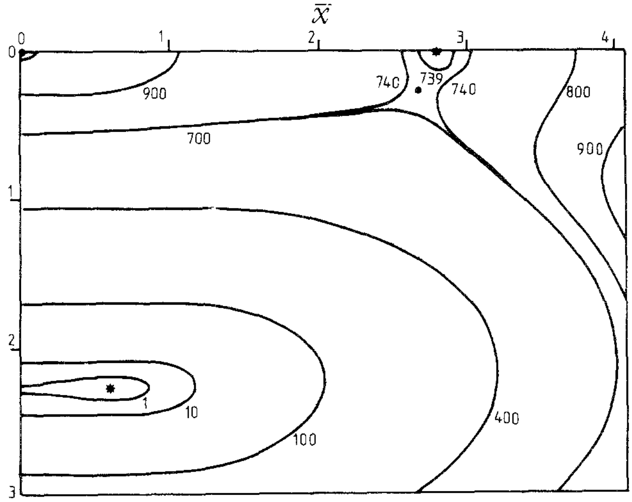

Fig. 5. Liapunov function for $\alpha = 2$, $\theta = 0.4$, $A = 3$, displaying coexistence of ferromagnetic and mixed phases.

bifurcation points in Figs. 2 and 3: after a short transient, initial probability densities are asymptotically close to a function of the form (2.1).[2,12] Near a bifurcation point, the order parameter $\bar{x}$ and/or q then slowly evolve toward their stable values. The detailed evolution can be obtained by adapting the multiscale method of ref. 12.

In Fig. 4 coexistence between spin-glass and ferromagnetic phases is visualized. At a smaller θ (Fig. 5) the spin-glass phase has lost its stability in favor of a mixed phase, which coexists with the ferromagnetic one. We have not depicted the cases in Figs. 2 and 3 where only one minimum of H_0 is present.

3. DISCUSSION

By adopting a soft-spin version of van Hemmen's spin glass model,[6] we have analyzed the effect of random parameters in the simple double-well Desai–Zwanzig model.[1]

As in the original model,[6] a reduced description holds in the thermodynamic limit $N \to \infty$. The bifurcation diagrams for stationary probability densities are qualitatively similar to those corresponding to hard spins.[2] In particular, the spin-glass phase is never stable unless the randomness is large enough ($J > J_0$). Dynamic relaxation to stable phases is exponential for close enough initial distributions, as is the case for Glauber dynamics of the hard-spin model.[9] A perturbation study (similar to those in refs. 1, 2, and 12) shows that the critical exponents for the different bifurcations between stationary probability densities are the classical ones (as in the double-well potential[1]). This contrast with models with stable time-dependent probability densities, where the presence of random parameters strongly alters the critical exponents and the phases themselves.[5]

The stability of the different phases in Eqs. (1.5) is analyzed with the help of a Liapunov function first derived by Shiino[11] for the Desai and Zwanzig model. Assuming that the only attractors with a time-independent Liapunov function (2.5)–(2.6) are stationary solutions of (1.5), some global stability results follow. In particular, when only one phase in the bifurcation diagrams is linearly stable, all initial probability densities will evolve toward it.

Phase coexistence between the ferromagnetic and the spin-glass or mixed phases is found if $J > J_0$ (enough randomness). Thus, metastability and hysteresis are possible. The level curves of the Liapunov function shown in Fig. 4 suggest that the basin of attraction of the spin-glass phase is larger than that of the ferromagnetic phase. For a smaller θ, the basin corresponding to the ferromagnetic phase is wider than before (Fig. 5), but it is still smaller than that of the mixed phase which branched off from the spin glass solution.

The initial conditions for (1.5) are functions (probability densities), while the visualization in Figs. 4 and 5 presupposes that the only relevant information about the initial data are the values of $\bar{x}$ and q. This turns out to be the case close enough to the stationary densities and near bifurcation points. Comparison with Brownian simulation is necessary to check the further validity of our visualization and to ultimately assess the usefulness of the Liapunov function. This task will be undertaken in a future publication.

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